AN EXTENSION OF A TABLE OF ABSORPTION FOR ELSASSER BANDS*

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Elsasser [1] formulated the absorption by a band consisting of Lorentz-shaped lines equally spaced and of equal strengths. He found the mean absorption over a spectral interval D, the spacing between the lines, and through a homogeneous gas of path length u, to be

$$A=\sinh \beta\int_0^y \exp(-y \cosh \beta)\, J_0(iy)dy \tag{1}$$

where $\beta=2\pi\alpha/D$

and $y=(Su)/(D \sinh \beta)$

in which α is the half width of the lines at half maximum and S is the strength (integrated absorption coefficient) of an individual line.

Kaplan [2] has shown that equation (1) may be represented by

$$A=\sinh \beta \exp(-y) \left[J_0(iy)\sum_{n=1}^{\infty}a_n y^n+i\, J_1(iy)\sum_{n=1}^{\infty}b_n y^n\right] \tag{2}$$

in which $a_1=1$

$$b_1=-\beta/\sinh \beta, \quad \cosh \beta<3$$

$$nb_{n+1}=b_n+a_n$$

$$na_n=a_{n-1}+b_{n-1}+(-1)^{n+1}\, c^{n-1}/(n-1)!$$

where $c=\cosh \beta-1$

From this expansion Kaplan prepared a table of fractional absorption, in which β ranges from 0.01 to 1.0 and y from 0 to 40.

Radiative transfer problems arising in meteorology require an extension of the table. In the present work, therefore, the series solution of Kaplan has been employed to carry the table to the limits normally encountered in the atmosphere. Calculations were performed for $\beta=1.0$, 0.1, 0.01, and 0.001. As many as 60 decimal places were required in the evaluation of the coefficients. Exponential functions of the desired accuracy were obtained from Van Orstrand's [3] tables.

The ranges of y were 0 to 150,000 or to the point where A became unity to four decimal places. For intermediate values of β, an interpolation scheme was devised to determine the difference between the correct values of A and the error function approximation,

$$A=\operatorname{erf}\,(\beta\sqrt{y/2}\,) \tag{3}$$

Table 1 shows the results of this work. The calculated values of A are exact to the accuracy shown. The interpolated values should be correct to within 0.0001 in most cases. All values of A lying outside this table, except for $\beta>1.0$, can be calculated precisely to at least four decimal places from the error function approximation. Kaplan's table has been included as a part of table 1.

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REFERENCES

1. W. M. Elsasser, "Mean Absorption and Equivalent Absorption Coefficients," *Physical Review*, vol. 54, No. 2, July 15, 1938, pp. 126–129.
2. L. D. Kaplan, "Regions of Validity of Various Absorption-Coefficient Approximations," *Journal of Meteorology*, vol. 10, No. 2, April 1953, pp. 100–104.
3. C. E. van Orstrand, "Tables of the Exponential Function and of the Circular Sine and Cosine to Radian Argument," *Memoirs of the National Academy of Sciences*, vol. XIV, Fifth Memoir, 1925, 79 pp.

TABLE 1.—*Fractional absorption by an Elsasser band, with arguments y and β*

$y \backslash \beta$	.0001	.0002	.0003	.0004	.0006	.0008	.001	.002	.003	.004	.006	.008
.02	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002
.04	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002	.0003
.06	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002	.0004	.0005
.08	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0002	.0002	.0003	.0005	.0006
.10	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0004	.0006	.0008
.15	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0008	.0011
.2	.0000	.0000	.0000	.0001	.0001	.0002	.0002	.0004	.0005	.0007	.0010	.0014
.3	.0000	.0000	.0001	.0001	.0002	.0002	.0003	.0006	.0008	.0011	.0015	.0022
.4	.0000	.0001	.0001	.0001	.0002	.0003	.0003	.0007	.0010	.0013	.0020	.0027
.5	.0000	.0001	.0001	.0002	.0002	.0003	.0004	.0008	.0012	.0015	.0024	.0032
.6	.0001	.0001	.0001	.0002	.0003	.0004	.0005	.0009	.0014	.0018	.0027	.0037
.7	.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0010	.0015	.0020	.0030	.0041
.8	.0001	.0001	.0002	.0002	.0003	.0005	.0006	.0011	.0016	.0022	.0034	.0045
.9	.0001	.0002	.0002	.0002	.0004	.0005	.0006	.0013	.0019	.0024	.0036	.0049
1.0	.0001	.0002	.0002	.0003	.0004	.0005	.0007	.0014	.0020	.0026	.0040	.0053
1.5	.0001	.0002	.0003	.0004	.0005	.0007	.0009	.0018	.0026	.0035	.0053	.0070
2	.0001	.0002	.0003	.0005	.0006	.0008	.0011	.0020	.0031	.0042	.0063	.0083
3	.0001	.0003	.0004	.0006	.0008	.0011	.0013	.0027	.0040	.0052	.0079	.0106
4	.0002	.0003	.0005	.0006	.0010	.0012	.0015	.0031	.0046	.0062	.0093	.0124
5	.0002	.0004	.0005	.0007	.0011	.0014	.0017	.0035	.0053	.0069	.0105	.0140
6	.0002	.0004	.0006	.0008	.0012	.0016	.0019	.0038	.0058	.0076	.0115	.0153
7	.0002	.0004	.0006	.0008	.0013	.0017	.0021	.0041	.0062	.0082	.0125	.0166
8	.0002	.0005	.0007	.0009	.0014	.0018	.0022	.0044	.0067	.0089	.0134	.0179
9	.0002	.0005	.0007	.0010	.0014	.0019	.0024	.0048	.0071	.0095	.0142	.0190
10	.0003	.0005	.0008	.0010	.0015	.0020	.0025	.0050	.0075	.0100	.0149	.0200
11	.0003	.0005	.0008	.0011	.0016	.0021	.0026	.0053	.0079	.0105	.0157	.0210
12	.0003	.0006	.0008	.0011	.0017	.0022	.0027	.0055	.0083	.0110	.0165	.0220
13	.0003	.0006	.0009	.0012	.0017	.0023	.0028	.0058	.0086	.0114	.0171	.0228
14	.0003	.0006	.0009	.0012	.0018	.0024	.0030	.0060	.0090	.0119	.0178	.0237
15	.0003	.0006	.0009	.0012	.0019	.0025	.0031	.0062	.0093	.0123	.0184	.0245
16	.0003	.0006	.0010	.0013	.0019	.0026	.0032	.0064	.0096	.0128	.0191	.0253
20	.0004	.0007	.0011	.0014	.0021	.0028	.0035	.0071	.0107	.0143	.0213	.0284
30	.0004	.0009	.0013	.0018	.0026	.0035	.0044	.0087	.0131	.0175	.0261	.0349
40	.0005	.0010	.0015	.0020	.0030	.0040	.0050	.0101	.0151	.0202	.0302	.0403
50	.0006	.0011	.0017	.0022	.0034	.0045	.0056	.0113	.0169	.0226	.0337	.0450
60	.0006	.0012	.0019	.0025	.0037	.0050	.0062	.0124	.0185	.0247	.0370	.0493
70	.0007	.0013	.0020	.0027	.0040	.0053	.0067	.0133	.0200	.0267	.0399	.0533
80	.0007	.0014	.0021	.0028	.0043	.0057	.0071	.0143	.0214	.0285	.0427	.0569
90	.0008	.0015	.0023	.0030	.0045	.0060	.0076	.0151	.0227	.0303	.0454	.0604
100	.0008	.0016	.0024	.0032	.0048	.0064	.0080	.0160	.0239	.0319	.0478	.0637
150	.0010	.0020	.0029	.0039	.0059	.0078	.0098	.0195	.0293	.0391	.0586	.0779
200	.0011	.0022	.0034	.0045	.0068	.0090	.0113	.0226	.0338	.0451	.0676	.0900
300	.0014	.0028	.0042	.0055	.0083	.0111	.0138	.0276	.0414	.0552	.0828	.1101
400	.0016	.0032	.0048	.0064	.0096	.0128	.0160	.0319	.0478	.0638	.0955	.1271
500	.0018	.0036	.0054	.0071	.0107	.0143	.0178	.0357	.0535	.0713	.1037	.1420
600	.0020	.0039	.0059	.0078	.0117	.0156	.0195	.0391	.0586	.0780	.1168	.1554
700	.0021	.0042	.0063	.0084	.0127	.0169	.0211	.0422	.0633	.0843	.1261	.1676
800	.0022	.0045	.0068	.0090	.0135	.0180	.0226	.0451	.0676	.0901	.1348	.1790
900	.0024	.0048	.0072	.0096	.0144	.0192	.0239	.0478	.0717	.0955	.1428	.1897
1000	.0025	.0050	.0076	.0101	.0151	.0202	.0252	.0504	.0756	.1007	.1505	.1997
1500	.0031	.0062	.0093	.0124	.0185	.0247	.0309	.0617	.0925	.1231	.1838	.2433
2000	.0036	.0071	.0107	.0143	.0214	.0285	.0357	.0713	.1067	.1420	.2116	.2795
3000	.0044	.0087	.0131	.0175	.0262	.0350	.0437	.0872	.1305	.1734	.2576	.3387
4000	.0050	.0101	.0151	.0202	.0303	.0404	.0504	.1006	.1505	.1997	.2957	.3871
5000	.0056	.0113	.0169	.0226	.0338	.0451	.0564	.1125	.1680	.2227	.3286	.4284
6000	.0062	.0124	.0185	.0247	.0371	.0494	.0617	.1231	.1837	.2433	.3579	.4645
7000	.0067	.0133	.0200	.0267	.0400	.0534	.0667	.1329	.1982	.2621	.3843	.4967
8000	.0071	.0143	.0214	.0285	.0428	.0570	.0713	.1420	.2116	.2795	.4085	.5257
9000	.0076	.0151	.0227	.0303	.0454	.0605	.0756	.1505	.2240	.2957	.4308	.5521
10000	.0080	.0160	.0239	.0319	.0478	.0638	.0797	.1585	.2358	.3108	.4515	.5763
2×10^4	.0113	.0226	.0338	.0451	.0676	.0901	.1125	.2227	.3286	.4284	.6038	.7421
3×10^4	.0138	.0276	.0414	.0552	.0828	.1102	.1375	.2710	.3967	.5116	.7013	.8341
4×10^4	.0160	.0319	.0478	.0638	.0955	.1271	.1585	.3108	.4515	.5763	.7699	.8904
5×10^4	.0178	.0357	.0535	.0713	.1067	.1420	.1769	.3453	.4977	.6289	.8203	.9264
6×10^4	.0196	.0391	.0586	.0780	.1168	.1554	.1935	.3758	.5376	.6728	.8584	.9500
7×10^4	.0211	.0422	.0633	.0843	.1261	.1676	.2087	.4033	.5726	.7101	.8876	.9657
8×10^4	.0226	.0451	.0676	.0901	.1348	.1790	.2227	.4284	.6039	.7421	.9103	.9764
9×10^4	.0239	.0478	.0717	.0955	.1428	.1897	.2358	.4515	.6319	.7699	.9281	.9836
10×10^4	.0252	.0504	.0756	.1006	.1505	.1997	.2482	.4729	.6572	.7941	.9422	.9886
15×10^4	.0309	.0617	.0925	.1231	.1838	.2433	.3015	.5614	.7547	.8787	.9799	.9980

$y \backslash \beta$	.01	.02	.03	.04	.06	.08
.02	.0002	.0004	.0006	.0008	.0012	.0016
.04	.0004	.0008	.0012	.0016	.0024	.0031
.06	.0006	.0012	.0017	.0023	.0035	.0047
.08	.0008	.0015	.0023	.0031	.0046	.0062
.10	.0010	.0019	.0029	.0038	.0057	.0076
.15	.0014	.0028	.0042	.0056	.0084	.0112
.2	.0018	.0036	.0055	.0073	.0109	.0146
.3	.0026	.0052	.0078	.0104	.0157	.0209
.4	.0033	.0067	.0100	.0133	.0200	.0267
.5	.0040	.0080	.0120	.0160	.0240	.0321
.6	.0046	.0093	.0139	.0185	.0278	.0370
.7	.0052	.0104	.0156	.0208	.0312	.0417
.8	.0057	.0115	.0172	.0230	.0345	.0460
.9	.0063	.0125	.0188	.0250	.0375	.0501
1.0	.0067	.0135	.0202	.0269	.0404	.0539
1.5	.0088	.0176	.0264	.0352	.0528	.0703
2	.0105	.0209	.0314	.0419	.0628	.0837
3	.0132	.0264	.0396	.0527	.0791	.1053
4	.0154	.0309	.0463	.0617	.0924	.1230
5	.0174	.0347	.0521	.0694	.1040	.1383
6	.0191	.0382	.0573	.0764	.1144	.1521
7	.0207	.0414	.0621	.0827	.1238	.1646
8	.0222	.0444	.0665	.0886	.1326	.1762
9	.0236	.0472	.0707	.0942	.1409	.1871
10	.0249	.0498	.0746	.0994	.1486	.1973
11	.0262	.0523	.0783	.1043	.1560	.2069
12	.0273	.0547	.0819	.1091	.1630	.2161
13	.0285	.0569	.0853	.1136	.1697	.2249
14	.0296	.0591	.0886	.1179	.1761	.2333
15	.0306	.0612	.0917	.1221	.1823	.2414
16	.0317	.0633	.0948	.1261	.1882	.2492
20	.0354	.0708	.1061	.1411	.2104	.2779
30	.0435	.0869	.1300	.1727	.2566	.3374
40	.0503	.1003	.1500	.1991	.2948	.3861
50	.0562	.1122	.1676	.2221	.3278	.4275
60	.0616	.1228	.1834	.2428	.3572	.4638
70	.0666	.1327	.1979	.2617	.3837	.4960
80	.0712	.1418	.2113	.2790	.4080	.5252
90	.0755	.1503	.2238	.2953	.4303	.5517
100	.0796	.1583	.2355	.3105	.4511	.5759
150	.0974	.1932	.2864	.3755	.5372	.6724
200	.1124	.2226	.3284	.4282	.6037	.7420
300	.1374	.2709	.3966	.5115	.7013	.8342
400	.1585	.3109	.4514	.5763	.7699	.8904
500	.1769	.3453	.4977	.6289	.8203	.9263
600	.1935	.3758	.5376	.6729	.8584	.9501
700	.2083	.4033	.5727	.7101	.8876	.9658
800	.2227	.4284	.6039	.7422	.9103	.9764
900	.2358	.4515	.6319	.7699	.9282	.9837
1000	.2481	.4729	.6572	.7942	.9423	.9886
1500	.3014	.5614	.7548	.8787	.9799	.9981
2000	.3453	.6289	.8203	.9264	.9927	.9997
3000	.4161	.7267	.8996	.9716	.9990	1.0000
4000	.4723	.7941	.9422	.9886	.9998	
5000	.5205	.8427	.9661	.9953	1.0000	
6000	.5614	.8787	.9799	.9981		
7000	.5972	.9058	.9879	.9992		
8000	.6289	.9264	.9927	.9996		
9000	.6572	.9422	.9956	.9998		
10000	.6827	.9545	.9973	.9999		
2×10^4	.8427	.9953	1.0000	1.0000		
3×10^4	.9167	.9995				
4×10^4	.9545	.9999				
5×10^4	.9747	1.0000				
6×10^4	.9857					
7×10^4	.9918					
8×10^4	.9953					
9×10^4	.9973					
10×10^4	.9984					
15×10^4	.9999					

$y \backslash \beta$	.10	.2	.3	.4	.5	.6	.7	.8	.9	1.0	β/y
.02	.0020	.0040	.0060	.0081	.0103	.0126	.0150	.0175	.0202	.0231	.02
.04	.0039	.0079	.0119	.0161	.0204	.0249	.0296	.0346	.0399	.0456	.04
.06	.0058	.0117	.0177	.0239	.0302	.0369	.0439	.0512	.0590	.0674	.06
.08	.0077	.0155	.0234	.0315	.0399	.0486	.0578	.0674	.0776	.0885	.08
.10	.0095	.0192	.0289	.0390	.0493	.0601	.0714	.0832	.0957	.1090	.10
.15	.0140	.0281	.0424	.0570	.0720	.0876	.1039	.1209	.1388	.1577	.15
.2	.0182	.0365	.0551	.0741	.0936	.1137	.1346	.1563	.1791	.2029	.2
.3	.0261	.0524	.0790	.1060	.1335	.1618	.1909	.2210	.2520	.2841	.3
.4	.0334	.0669	.1007	.1350	.1698	.2052	.2414	.2783	.3161	.3548	.4
.5	.0401	.0803	.1208	.1615	.2028	.2445	.2867	.3295	.3728	.4165	.5
.6	.0463	.0927	.1392	.1860	.2330	.2802	.3277	.3754	.4231	.4707	.6
.7	.0521	.1042	.1563	.2085	.2607	.3129	.3649	.4167	.4680	.5185	.7
.8	.0575	.1149	.1723	.2295	.2864	.3429	.3988	.4540	.5081	.5609	.8
.9	.0626	.1250	.1872	.2489	.3101	.3705	.4299	.4879	.5443	.5986	.9
1.0	.0673	.1345	.2011	.2672	.3323	.3961	.4584	.5188	.5769	.6322	1.0
1.5	.0878	.1749	.2605	.3437	.4239	.5004	.5724	.6394	.7007	.7592	1.5
2	.1045	.2076	.3077	.4036	.4940	.5778	.6541	.7222	.7819	.8328	2
3	.1314	.2596	.3817	.4949	.5973	.6872	.7639	.8273	.8780	.9170	3
4	.1534	.3016	.4398	.5642	.6720	.7619	.8338	.8888	.9291	.9570	4
5	.1725	.3374	.4882	.6197	.7292	.8158	.8808	.9270	.9579	.9772	5
6	.1895	.3689	.5296	.6657	.7744	.8560	.9135	.9514	.9746	.9877	6
7	.2050	.3972	.5659	.7045	.8108	.8866	.9367	.9674	.9845	.9933	7
8	.2193	.4229	.5981	.7377	.8405	.9102	.9534	.9779	.9908	.9963	8
9	.2327	.4465	.6269	.7664	.8650	.9285	.9655	.9853	.9943	.9981	9
10	.2452	.4683	.6529	.7914	.8854	.9429	.9744	.9899	.9964	.9989	10
11	.2571	.4887	.6765	.8133	.9024	.9542	.9807	.9928	.9975	.9992	11
12	.2683	.5077	.6980	.8325	.9168	.9631	.9852	.9947	.9982	.9994	12
13	.2790	.5256	.7177	.8496	.9288	.9700	.9886	.9961	.9988	.9996	13
14	.2893	.5424	.7358	.8646	.9391	.9757	.9913	.9971	.9991	.9998	14
15	.2992	.5583	.7525	.8780	.9478	.9802	.9932	.9978	.9993	.9999	15
16	.3086	.5734	.7679	.8900	.9552	.9842	.9954	.9985	.9997	1.0000	16
20	.3434	.6267	.8192	.9265	.9752	.9930	.9983	.9996	1.0000		20
30	.4147	.7255	.8995	.9719	.9940	.9990	.9997	1.0000			30
40	.4718	.7934	.9423	.9890	.9986	.9998	1.0000				40
50	.5196	.8424	.9665	.9955	.9996	1.0000					50
60	.5606	.8785	.9801	.9981	.9999						60
70	.5966	.9056	.9881	.9991	1.0000						70
80	.6284	.9263	.9928	.9996							80
90	.6568	.9423	.9957	.9999							90
100	.6822	.9547	.9974	1.0000							100
150	.7791	.9857	.9998								150
200	.8426	.9955	1.0000								200
300	.9168	.9995									300
400	.9544	.9999									400
500	.9747	1.0000									500
600	.9857										600
700	.9919										700
800	.9950										800
900	.9973										900
1000	.9984										1000
1500	.9999										1500
2000	1.0000										2000
3000											3000
4000											4000
5000											5000
6000											6000
7000											7000
8000											8000
9000											9000
10000											10000
2×10^4											2×10^4
3×10^4											3×10^4
4×10^4											4×10^4
5×10^4											5×10^4
6×10^4											6×10^4
7×10^4											7×10^4
8×10^4											8×10^4
9×10^4											9×10^4
10×10^4											10×10^4
15×10^4											15×10^4